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APPROXIMATE ANALYTICAL SOLUTIONS OF SMART COMPOSITE MINDLIN BEAMS

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In this paper, a refined theory and approximate analytical solutions of laminated composite beams with piezoelectric laminae are developed. The equations of motion of the theory are developed using an energy principle. This formulation is based on linear piezoelectricity and Mindlin lamination theory, and includes the coupling between mechanical deformations and the charge equations of electrostatics. The approximate analytical solutions, using software package MATLAB and MATHEMATICA, are to study the effectiveness of piezoelectric sensors and actuators in actively controlling the transverse response of smart laminated beams. A main feature of this work is that it introduces the displacement potential function to simplify the governing equation. A new assumption of harmonic vibration and the transformation method of complex numbers are introduced. It can be used in differential equations that include both items of the functions sin and cosine, and the odd-order differential coefficient. The behaviour of the output voltage from the sensor layer and the input voltage acting on the actuator layer is also studied. Graphical results are presented to demonstrate the ability of a closed-loop system to actively control the vibration of laminated beams. The present method has a general application in this field of study.

1. INTRODUCTION

Due to the increasing demands of high structural performance requirements, the study of embedded or surface-mounted piezoelectric materials in structures has received considerable attention in recent years. Smart structures technology featuring a network of sensors and actuators, real-time control capabilities, computational capabilities and host material will have a tremendous impact upon the design, development and manufacture of the next generation of products in diverse industries. The idea of applying smart materials to mechanical and structural systems has been studied by researchers in various disciplines. Polyvinylidene fluoride (PVDF) was initially discovered by Kawai in 1969 [1]. Raw polymetric PVDF (α-phase) is an electrical insulator and it does not have any intrinsic piezoelectric properties. If the raw material is polarized during the manufacturing process, PVDF transforms to β-phase—a tough and flexible semi-crystalline material and it can be made to strain in either one or two directions in the film plane. Since β-phase PVDF possesses a strong direct piezoelectric effect, it has been used in many transducer applications: e.g., sonar, medical ultrasonic equipment, robot tactile sensors, acoustic pick-ups, forces and strains gauges, etc. Due to its distinct characteristics, such as flexibility, durability, manufacturability, etc., PVDF is an ideal material for the distributed sensing and vibration suppression/control of distributed parameter systems (e.g., beams, plates, shells, etc.).

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In order to utilize the strain-sensing and actuating properties of piezoelectric materials, the interaction between the structure and SSA (strain sensing and actuating) material must be well understood. There have been many theories and models proposed for analysis of laminated composite beams and plates containing active and passive piezoelectric layers. Bailey and Hubbard [2] designed a distributed-parameter actuator and control theory. They used the angular velocity at the tip of cantilever isotropic beam with constant-gain and constant-amplitude negative velocity algorithms and experimentally achieved vibration control. Hanagud *et al.* ([3]) presented a procedure, combining theory and experiments, to quantify the effects of an active feedback system on the damping matrix of an isotropic beam. Mechanical models for studying the interaction of piezoelectric patches surface-mounted to beams have been developed by Crawley and de Luis [4], Im and Atluri [5], and Chandra and Chopra [6]. The study presented here is different from these in that laminated beams containing piezoelectric laminae are studied. The strain-sensing and actuating (SSA) lamina can offer both discrete effects similar to patches as well as distributed effect. Gerhold and Rocha [7] used piezoelectric sensor/driver pairs that are collocated equidistant from the neutral axis for the active vibration control of free–free isotropic beams using constant-gain feedback control. They neglected the effect of piezoelectric elements on the mass and stiffness properties of the beam element. The modelling aspects of laminated plates incorporating the piezoelectric property of materials have been reported by Lee [8] and Crawley and Lazarus [9]. Wang and Rogers [10] used the assumptions of classical lamination theory combined with inclusion of the effects of spatially distributed, small-size induced strain actuators embedded at any location of the laminate. Lee [8] also derived a theory for laminated piezoelectric plates, where the linear piezoelectric constitutive equations were the only source of coupling between the electric field and the mechanical displacement field. Pai *et al.* [11] presented a geometrically non-linear plate theory for the analysis of composite plates with distributed piezoelectric laminate. However, their model does not include the charge equations of electrostatics. These models are based on classical laminated plate theory, which neglects the transverse shear effects. However; the effects of transverse shear stresses are important in composite fibre-reinforced materials because the interlaminar shear module is usually much smaller than the in-plane Young's module. In contrast, Tzou and Gadre [12] derived equations of motion for laminated shells with piezoelectric layers based upon Love's first-approximation shell theory and Hamilton's principle. At that time, they did not include the charge equations in the model. Later, Tzou and Zhong [13] derived governing equations for piezoelectric shells using first order shear deformation theory and included the charge equations of electrostatics. A finite element model for the active vibration control of laminated plate based on first order shear deformation theory has been presented by Chandrashekhara and Agarwal [14]. An overview of recent developments in the area of sensing and control of structures by piezoelectric materials has been reported in Rao and Sunar [15]. Recently, the issue of the feedback control gain of smart composite structures has also been discussed by Sun and Huang [16].

Compared with the analysis of laminated plates without piezoelectric layers, the work reported in the area of fibre-reinforced composite beams with piezoelectric layers is still quite limited, especially for active vibration control of composite beams with piezoelectric laminae. Also, there are quite extensive studies carried out using the finite element method. The different finite element models for smart laminated composite beams have been well established. However, for the analytical solution or exact solution, very few studies concentrated on this research area. The present work is to develop a set of governing equations for laminated composite beams with piezoelectric laminae using Hamilton's principle by introducing the electric potential function. The approximate analytical solutions of smart laminated beams with piezoelectric laminae based on a first order shear deformation theory (Mindlin plate theory [17]) is to be derived by using the special method. The behaviour of output voltage from sensor and input voltage acting on actuator will be also discussed. The major goal of this paper is to investigate the vibrational characteristic of smart composite beams and an accurate solution for vibration control of smart beams is presented. It can be used in differential equations that include both items of the functions sin and cosine, and the odd-order differential coefficient. The present method has a general application in this field of study.

2. MATHEMATICAL FORMULATIONS

Constitutive equations are developed for laminated beams with piezoelectric sensor and actuator layers. The shear deformation effect is incorporated in the formulation using Mindlin plate theory. Consider a smart laminated beam having length L, width b, and thickness h (Figure 1). The electric field is applied through the thickness of the piezoelectric material. The constitutive equations including piezoelectric effects with respect to the plane (laminate) co-ordinates is (x, y, z), are [18]

$$\{\sigma\}_k = [\bar{Q}]_k\{\varepsilon\}_k - [\bar{e}]_k^{\mathrm{T}}\{E\}_k, \tag{1}$$

$$\{D\}_k = [\bar{e}]_k\{\varepsilon\}_k + [\bar{g}]_k\{E\}_k, \tag{2}$$

where $\{D\}$, $\{E\}$, $\{\varepsilon\}$ and $\{\sigma\}$ are the electric displacement, electric field, strain and stress vectors, and $[\bar{Q}]$, $[\bar{e}]$, $[\bar{g}]$ are the elasticity, piezoelectric and permittivity constant matrices, respectively. $[\bar{e}]^{\mathrm{T}}$ is defined as the transpose of $[\bar{e}]$. Equation (1) describes the inverse piezoelectric effect and equation (2) describes the direct piezoelectric effect.

It is to be noted that the present model of a beam derives from a plate. The assumption at this point is that existing theories can be utilized. At this point in the development of composite technology, simplifications of plate theory appear to offer the most feasible approaches from which to start. In the present case, the beam is a smart composite beam

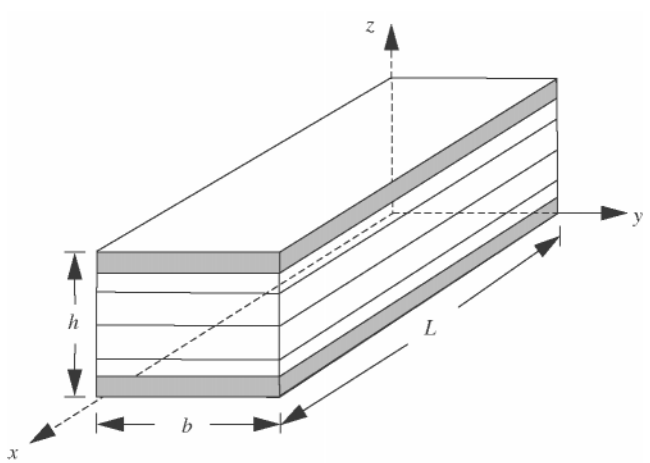

Figure 1. Laminated beam with integrated piezoelectric sensor and actuator. □ Non-piezoelectric layer; ▒, Piezoelectric layers.

model including the piezoelectric materials, which possess anisotropic properties. PVDF (polyvinylidene fluoride) and PZT (piezoceramics, such as lead zirconate titanates) are excellent candidates for the role of sensors and actuators. In this project, the PVDF is chosen as the material of both the sensors and actuators. Piezoelectric material layers are polarized in the thickness direction and exhibit transversely isotropic properties in the xy-plane. Considering piezoelectric materials while retaining the anisotropic behaviour of the master structure, equation (1) can be written as [19]

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_k - \begin{bmatrix} 0 & 0 & \bar{e}_{31} \\ 0 & 0 & \bar{e}_{31} \\ \bar{e}_{25} & \bar{e}_{15} & 0 \\ \bar{e}_{15} & \bar{e}_{25} & 0 \\ 0 & 0 & 0 \end{bmatrix}_k \begin{Bmatrix} 0 \\ 0 \\ E_x \end{Bmatrix}_k \tag{3}$$

where $\bar{e}_{31} = e_{31}$, $\bar{e}_{15} = e_{15}(\cos^2\theta - \sin^2\theta)$ and $\bar{e}_{25} = -2e_{15}\sin\theta\cos\theta$.

Thus, because of the above statement, for a beam problem one can use $\sigma_y = \tau_{yz} = \tau_{xy} = 0$ while assuming that $\varepsilon_y \neq \gamma_{yz} \neq \gamma_{xy} \neq 0$, to obtain the following reduced constitutive equations of smart composite beams from equation (3) [20].

$$\begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix}_k = \begin{bmatrix} \tilde{Q}_{11} & 0 \\ 0 & \tilde{Q}_{55} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix}_k - \begin{Bmatrix} \tilde{e}_{31} \\ 0 \end{Bmatrix}_k E_z^k, \tag{4}$$

where

$$\begin{aligned} \tilde{Q}_{11} &= \bar{Q}_{11} + \frac{\bar{Q}_{16}\bar{Q}_{26} - \bar{Q}_{12}\bar{Q}_{66}}{\bar{Q}_{22}\bar{Q}_{66} - \bar{Q}_{26}^2}\bar{Q}_{12} + \frac{\bar{Q}_{12}\bar{Q}_{26} - \bar{Q}_{16}\bar{Q}_{22}}{\bar{Q}_{22}\bar{Q}_{66} - \bar{Q}_{26}^2}\bar{Q}_{16}, \\ \tilde{Q}_{55} &= \bar{Q}_{55} - \frac{\bar{Q}_{45}^2}{\bar{Q}_{44}}, \\ \tilde{e}_{31} &= \left(1 - \frac{\bar{Q}_{12}\bar{Q}_{66} - \bar{Q}_{16}\bar{Q}_{26}}{\bar{Q}_{22}\bar{Q}_{66} - \bar{Q}_{26}^2}\right)\bar{e}_{31}. \end{aligned} \tag{5}$$

The displacement field for the present composite beams based on Mindlin plate theory can be written as [21]

$$\begin{aligned} u_1(x, y, z, t) &= u_0(x, t) + z\Psi_0(x, t), \\ u_2(x, y, z, t) &= 0, \\ u_3(x, y, z, t) &= w\Psi_0\,(x, t), \end{aligned} \tag{6}$$

where it is assumed that the displacement for y direction is neglected and u_0, w_0 and ψ_0 are only functions of x-axis and time (t) in the present model of beam. The strain displacement relations of a laminated beam based on a first order shear deformation theory associated with the displacement field are given by

$$\varepsilon_x = \varepsilon_x^0 + z\kappa_x^1, \qquad \gamma_{xz} = \gamma_{xz}^0, \tag{7}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \qquad \kappa_x^1 = \frac{\partial \psi_0}{\partial x}, \qquad \gamma_{xz}^0 = \psi_0 + \frac{\partial w_0}{\partial x}. \tag{8}$$

In the present beam model, it is assumed that the bottom and top layers are sensor and actuator layers respectively. For sensor laminae, no external electric field is applied to this layer. Then, the electric field intensity for sensor is zero. Substituting equation (7) into equation (4), setting the sensor layer as $E_z^k = 0$ and integrating through the thickness, the stress resultants can be obtained as

$$\begin{Bmatrix} N_x \\ M_x \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} & 0 \\ \bar{B}_{11} & \bar{D}_{11} & 0 \\ 0 & 0 & \bar{A}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \kappa_x^1 \\ \gamma_{xz}^0 \end{Bmatrix} - \begin{Bmatrix} N_x^p \\ M_x^p \\ 0 \end{Bmatrix}, \tag{9}$$

where

$$\{\bar{A}_{11} \;\; \bar{B}_{11} \;\; \bar{D}_{11}\} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \tilde{Q}_{11}(1, z, z^2)\,\mathrm{d}z$$

$$\{\bar{A}_{55}\} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \bar{Q}_{55}\,\mathrm{d}z$$

$$\{N_x^p \;\; M_x^p\} = \int_{z_{n-1}}^{z_n} \tilde{e}_{31} E_z^k (1, z)\,\mathrm{d}z$$

and z_k is defined in Figure 2. Note that $z_0 = -h/2$, $z_n = h/2$.

According to Mitchell and Reddy [22], the electric potential variable Φ can be expressed as

$$\Phi(x, y, z, t) = f(z)\phi_0(x, y, t). \tag{10}$$

For the beam problem, the varying of y-axis is not considered because any physical variables are uniformly distributed through the Y direction. The electric field of smart composite beams can be written as

$$\Phi(x, z, t) = f(z)\phi_0(x, t). \tag{11}$$

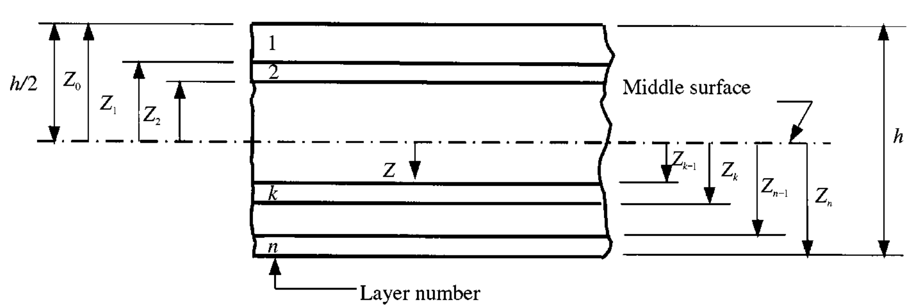

Figure 2. Geometry of an n-layered laminated beam.

Since the thickness of the piezoelectric layers is very thin, it is also assumed that the voltage is uniformly distributed through the thickness (Z direction) of the piezoelectric layers. That is $f(z) = 1$. Then the electric field intensity E_z^k can be expressed as

$$E_z^k = \phi_o(x, t)/h_p \tag{12}$$

where h_p is the thickness of the piezoelectric layers.

3. GOVERNING EQUATIONS

Hamilton's principle can be expressed mathematically as

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W)\,\mathrm{d}t = 0, \tag{13}$$

where $U = U_M + U_E$, T is the kinetic energy, U_M the strain energy, U_E the electrical field potential energy of piezoelectric layers, and W the work done by surface tractions and applied surface electrical charge density. In this project, the body forces are not considered. In equation (13), t_1 and t_2 are two arbitrary time variables and δ denotes the first variation. Starting with the first integral, it is assumed that each layer of the present composite beam model has the same vibration speed. The first variation kinetic energy can be expressed as

$$\delta T = \int_0^1 \left[\left(I_1 \frac{\partial u_0}{\partial t} + I_2 \frac{\partial \psi_0}{\partial t} \right) \frac{\partial \delta u_0}{\partial t} + I_1 \frac{\partial w_0}{\partial t} \frac{\partial \delta w_0}{\partial t} + \left(I_2 \frac{\partial u_0}{\partial t} + I_3 \frac{\partial \psi_0}{\partial t} \right) \frac{\partial \delta \psi_0}{\partial t} \right] b\,\mathrm{d}x, \tag{14}$$

where

$$(I_1, I_2, I_3) = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \rho_k (1, z, z^2)\,\mathrm{d}z.$$

The variation in strain energy is given by

$$\begin{aligned} \delta U_M &= \int_0^1 (N_x \delta \varepsilon_x^0 + M_x \delta \kappa_x^1 + Q_{xz} \gamma_{xz}^0) b\,\mathrm{d}x \\ &= \int_0^1 \left[N_x \frac{\partial \delta u_0}{\partial x} + Q_{xz} \frac{\partial \delta w_0}{\partial x} + M_x \frac{\partial \delta \psi_0}{\partial x} + Q_{xz} \delta \psi_0 \right] b\,\mathrm{d}x. \end{aligned} \tag{15}$$

Piezoelectric materials are polarized in the thickness direction and exhibit transversely isotropic properties in XY-plane. So for equation (2), only D_z is of interest here. Considering piezoelectric materials while retaining the anisotropic behaviour of the master structure, the constitutive equation (2) can be written as

$$D_z^k = \bar{e}_{31} \varepsilon_x + \bar{e}_{31} \varepsilon_y + \bar{g}_{33} E_z^k. \tag{16}$$

For the beam problem, equation (16) can be written as

$$D_z^k = \tilde{e}_{31} \varepsilon_x + \tilde{g}_{33} E_z^k, \tag{17}$$

where

$$\tilde{e}_{31} = \left(1 - \frac{\bar{Q}_{12}\bar{Q}_{66} - \bar{Q}_{16}\bar{Q}_{26}}{\bar{Q}_{22}\bar{Q}_{66} - \bar{Q}_{26}^2}\right)\bar{e}_{31}, \qquad \tilde{g}_{33} = \frac{\bar{Q}_{66}\bar{e}_{31}^2}{\bar{Q}_{22}\bar{Q}_{66} - \bar{Q}_{26}^2} + \bar{g}_{33}.$$

The first variation of the electrical field potential energy U_E is obtained as

$$\delta U_E = \int_0^1 G_z^k \delta\phi_0 b \,\mathrm{d}x, \tag{18}$$

where

$$G_z^k = \sum_{k=1}^{n} D_z^k \frac{1}{h_p} \,\mathrm{d}z.$$

The variation of the virtual work done by external surface force and the applied surface charge density can be expressed as

$$\delta W = \int_0^1 p\delta w_0 b \,\mathrm{d}x - \int_0^1 q_0 \delta\phi_0 b \,\mathrm{d}x, \tag{19}$$

where p is the surface traction and q_0 is the surface charge density applied to the present intelligent composite beam model respectively. Substituting equations (14), (15), (18) and (19) into Hamilton's principle (equation (13)) the governing equations for vibration of smart composite beams based on the MINDLIN plate theory can be expressed as

$$\delta u_0: \quad I_1\ddot{u}_0 + I_2\ddot{\psi}_0 - \frac{\partial N_x}{\partial x} = 0, \tag{20a}$$

$$\delta w_0: \quad I_1\ddot{w}_0 - \frac{\partial Q_{xz}}{\partial x} = p, \tag{20b}$$

$$\delta \psi_0: \quad I_2\ddot{u}_0 + I_3\ddot{\psi}_0 - \frac{\partial M_x}{\partial x} + Q_{xz} = 0, \tag{20c}$$

$$\delta \phi_0: \quad G_z^k + q_0 = 0 \tag{20d}$$

and equations (20) are subject to the following boundary conditions:

$$x = 0, \text{ and } x = l: \quad N_x = \tilde{N}_x, \textit{ or } u_0 = \tilde{u}_0,$$

$$x = 0, \text{ and } x = l: \quad Q_{xz} = \tilde{Q}_{xz}, \textit{ or } w_0 = \tilde{w}_0,$$

$$x = 0, \text{ and } x = l: \quad M_x = \tilde{M}_x, \textit{ or } \psi_0 = \tilde{\psi}_0$$

and "$\sim$" denotes the known value.

According to the coefficient G_z^k and equations (17) and (20d), the electrical field potential function can be expressed as

$$\phi_0(x, t) = -\frac{\tilde{e}_{31}G_1^k}{\tilde{g}_{33}G_3^k}\frac{\partial u_0}{\partial x} - \frac{\tilde{e}_{31}G_2^k}{\tilde{g}_{33}G_3^k}\frac{\partial \psi_0}{\partial x} + \phi_A(x, t), \tag{21}$$

where $\phi_A(x, t)$ is the input control electric potential voltage acting on the actuator layer. If the sensing information is required, the electrical potential can be recovered by

$$\phi_0(x, t) = \alpha_1 \frac{\partial u_0}{\partial x} + \alpha_2 \frac{\partial \psi_0}{\partial x} + \phi_A(x, t), \tag{22}$$

where

$$\alpha_1 = \frac{\tilde{e}_{31} G_1^k}{\tilde{g}_{33} G_3^k}, \qquad \alpha_2 = -\frac{\tilde{e}_{31} G_2^k}{\tilde{g}_{33} G_3^k}.$$

Note that $\phi_A(x, t)$ is usually zero in the piezoelectric sensor layer. Thus, the piezoelectric sensor electrical potential output is estimated by

$$\phi_S(x, t) = \alpha_{1S} \frac{\partial u_0}{\partial x} + \alpha_{2S} \frac{\partial \psi_0}{\partial x}. \tag{23}$$

In the active vibration control application the electric force term can be regarded as the feedback control force. The piezoelectric actuator electrical potential input in terms of the output signal from the piezoelectric sensor layer can be written as

$$\phi_A(x, t) = -G_i \left(\alpha_{1A} \frac{\partial^2 u_0}{\partial x \partial t} + \alpha_{2A} \frac{\partial^2 \psi_0}{\partial x \partial t} \right), \tag{24}$$

where the negative velocity feedback control method is implemented and G_t is the feedback control gain. The symbols α_{jS} and $\alpha_{jA} = (j = 1, 2)$ are the relative coefficients of sensor laminae and actuator laminae, respectively.

Substituting equations (9), (21) and (24) into equations (20a–c) and using the derivative operator forms, the governing equations can be written in a simple form in terms of the mechanical and piezoelectric resultants as

$$\delta u_0: \quad L_{11} u_0 + L_{12} w_0 + L_{13} \psi_0 = 0, \tag{25a}$$

$$\delta w_0: \quad L_{21} u_0 + L_{22} w_0 + L_{23} \psi_0 = p, \tag{25b}$$

$$\delta \psi_0: \quad L_{31} u_0 + L_{32} w_0 + L_{33} \psi_0 = 0, \tag{25c}$$

where L_{ij} are the derivative operators given by

$$L_{11} = -a \frac{\partial^2}{\partial x^2} + I_1 \frac{\partial^2}{\partial t^2} + G_i \alpha_1 \frac{\partial^3}{\partial x^2 \partial t}, \qquad L_{12} = L_{21} = 0,$$

$$L_{13} = -b \frac{\partial^2}{\partial x^2} + I_2 \frac{\partial^2}{\partial t^2} + G_i \alpha_2 \frac{\partial^3}{\partial x^2 \partial t},$$

$$L_{22} = -I_1 \frac{\partial^2}{\partial t^2} - \bar{A}_{55} \frac{\partial^2}{\partial x^2}, \quad L_{23} = -\bar{A}_{55} \frac{\partial}{\partial x},$$

$$L_{31} = -b\frac{\partial^2}{\partial x^2} + I_2\frac{\partial^2}{\partial t^2} + G_i\alpha_1\frac{\partial^3}{\partial x^2 \partial t}, \qquad L_{32} = \bar{A}_{55}\frac{\partial}{\partial x},$$

$$L_{33} = -d\frac{\partial^2}{\partial x^2} + I_3\frac{\partial^2}{\partial t^2} + G_i\alpha_2\frac{\partial^3}{\partial x^2 \partial t} + \bar{A}_{55}$$

and

$$a = \bar{A}_{11} + \beta_1, \qquad b = \bar{B}_{11} + \beta_2, \qquad d = \bar{D}_{11} + \beta_3,$$

$$\beta_1 = -\frac{\tilde{e}_{31}^2 G_1^{k^2}}{\tilde{g}_{33} G_3^k}, \qquad \beta_2 = -\frac{\tilde{e}_{31}^2 G_1^k G_2^k}{\tilde{g}_{33} G_3^k}, \qquad \beta_3 = -\frac{\tilde{e}_{31}^2 G_2^{k^2}}{\tilde{g}_{33} G_3^k}.$$

4. ANALYTICAL SOLUTIONS

Equations (25a) and (c) give

$$\begin{aligned} \Delta u_0 &= (-L_{13}L_{32} + L_{12}L_{33})\psi_0, \\ \Delta w_0 &= (-L_{11}L_{33} + L_{13}L_{31})\psi_0, \end{aligned} \tag{26}$$

where

$$\Delta = L_{11}L_{32} - L_{12}L_{31}.$$

Introducing the displacement potential function $F(x, t)$ and taking into account $L_{12} = L_{21} = 0$, gives

$$u_0 = L_1(F), \qquad w_0 = L_2(F), \qquad \psi_0 = L_3(F), \tag{27}$$

where

$$L_1 = -L_{13}L_{32}, \qquad L_2 = -L_{11}L_{33} + L_{13}L_{31}, \qquad L_3 = L_{11}L_{32},$$

Substituting equation (27) into equation (25b), the equation of the displacement potential function $F(x, t)$ is obtained as

$$L_{22}L_2(F) + L_{23}L_3(F) = p(x, t). \tag{28}$$

In this paper, it is assumed that the external exciting force has the feature of harmonic vibration with the following form [20]:

$$p(x, t) = p_1(x)\sin(\omega t) + p_2(x)\cos(\omega t) \tag{29}$$

and the displacement potential function $F(x, t)$ has the form

$$F(x, t) = K_1(x)\sin(\omega t) + K_2(x)\cos(\omega t) \tag{30}$$

Substituting equations (29) and (30) into equation (28) and separating the two variables of the field of space and time, the two coupled differentiation equations of functions $K_1(x)$ and $K_2(x)$ are given as [20]

$$b_6 \frac{\mathrm{d}^6 K_1(x)}{\mathrm{d}x^6} - b_4 \frac{\mathrm{d}^4 K_1(x)}{\mathrm{d}x^4} - b_2 \frac{\mathrm{d}^2 K_1(x)}{\mathrm{d}x^2} + b_1 K_1(x) - \left[b_7 \frac{\mathrm{d}^6 K_2(x)}{\mathrm{d}x^6} - b_5 \frac{\mathrm{d}^4 K_2(x)}{\mathrm{d}x^4} - b_3 \frac{\mathrm{d}^2 K_2(x)}{\mathrm{d}x^2} \right] = p_1(x), \tag{31a}$$

$$b_6 \frac{\mathrm{d}^6 K_2(x)}{\mathrm{d}x^6} - b_4 \frac{\mathrm{d}^4 K_2(x)}{\mathrm{d}x^4} - b_2 \frac{\mathrm{d}^2 K_2(x)}{\mathrm{d}x^2} + b_1 K_2(x) + \left[b_7 \frac{\mathrm{d}^6 K_1(x)}{\mathrm{d}x^6} - b_5 \frac{\mathrm{d}^4 K_1(x)}{\mathrm{d}x^4} - b_3 \frac{\mathrm{d}^2 K_1(x)}{\mathrm{d}x^2} \right] = p_2(x), \tag{31b}$$

where

$$b_1 = (a_9 - a_8\omega^2)\omega^4, \qquad b_2 = (a_6 - a_5\omega^2)\omega^2,$$

$$b_3 = (a_{10} - a_7\omega^2)\omega^3, \qquad b_4 = a_2\omega^2,$$

$$b_5 = a_4\omega^3, \qquad b_6 = a_1, \qquad b_7 = a_3\omega$$

and

$$a_1 = -\bar{A}_{55}b^2 + \bar{A}_{55}ad, \qquad a_2 = -\bar{A}_{55}(I_3 a + I_1 d - 2I_2 b) + I_1(b^2 - ad),$$

$$a_3 = -\bar{A}_{55}G_1(a\alpha_2 + d\alpha_1 - b\alpha_1 - b\alpha_2),$$

$$a_4 = -\bar{A}_{55}G_i(I_2\alpha_1 + I_2\alpha_2 - I_1\alpha_2 - I_3\alpha_1) + I_1 G_i(a\alpha_2 + d\alpha_1 - b\alpha_1 - b\alpha_2),$$

$$a_5 = -\bar{A}_{55}(I_2^2 - I_1 I_3) + I_1(I_3 a + I_1 d - 2I_2 b), \qquad a_6 = I_1 a\bar{A}_{55},$$

$$a_7 = I_1 G_i(I_2\alpha_1 + I_2\alpha_2 - I_1\alpha_2 - I_3\alpha_1), \qquad a_8 = I_1(I_2^2 - I_1 I_3),$$

$$a_9 = -\bar{A}_{55}I_1^2, \qquad a_{10} = -\bar{A}_{55}I_1 G_i\alpha_1.$$

The product of equation (31b) and imaginary unit i, and consequently the sum of the above product and equation (31a), allowed the reduced equation to become [20]

$$A_6 \frac{\mathrm{d}^6 K(x)}{\mathrm{d}x^6} + A_4 \frac{\mathrm{d}^4 K(x)}{\mathrm{d}x^4} + A_2 \frac{\mathrm{d}^2 K(x)}{\mathrm{d}x^2} + A_0 K(x) = \bar{p}(x), \tag{32}$$

where

$$A_6 = b_6 + \mathrm{i}b_7, \quad A_4 = -(b_4 + \mathrm{i}b_5), \qquad A_2 = -(b_2 + \mathrm{i}b_3)$$

$$A_0 = b_1, \qquad K(x) = K_1(x) + \mathrm{i}K_2(x), \qquad \bar{p}(x) = p_1(x) + \mathrm{i}p_2(x).$$

The solutions of equation (32) can be expressed as

$$K(x, t) = K_h(x, t) + K_p(x, t), \tag{33}$$

where $K_h(x, t)$ is the homogeneous solution and $K_p(x, t)$ is the particular solution.

Especially for uniformly distributed loads, $p_1(x)$ and $p_2(x)$ are constant, and consequently $\bar{p}(x)$ is also constant. Thus the particular solution of equation can be obtained as

$$K_p(x) = \bar{p}/A_0. \tag{34}$$

By using Software Package MATLAB, the homogeneous solution of the present beam model (32) can be written in the following form:

$$K_h(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x} + C_3 e^{k_3 x} + C_4 e^{k_4 x} + C_5 e^{k_5 x} + C_6 e^{k_6 x}, \tag{35}$$

where C_1–C_6 are the six constants of integration produced, which can be determined by using the boundary conditions as shown before. Here

$$k_1 = -\frac{1}{\sqrt{6A_6\xi}}[A_6\xi(-2A_4\xi + \xi^2 - 12A_2A_6 + 4A_4^2)]^{1/2},$$

$$k_2 = \frac{1}{\sqrt{6A_6\xi}}[A_6\xi(-2A_4\xi + \xi^2 - 12A_2A_6 + 4A_4^2)]^{1/2},$$

$$k_3 = \frac{1}{\sqrt{3A_6\xi}}[-A_4A_6\xi^2 - 9A_2A_4A_6^2 + 27A_0A_6^3 + 2A_4^3A_6 - 3\sqrt{3}A_6^2\zeta + 3A_2A_6^2\xi$$
$$- A_4^2A_6\xi + i\sqrt{3}(9A_2A_4A_6^2 - 27A_0A_6^3 - 2A_4^3A_6 + 3\sqrt{3}A_6^2\zeta + 3A_2A_6^2 - A_4^2A_6\xi)]^{1/2},$$

$$k_4 = -\frac{1}{\sqrt{3A_6\xi}}[-A_4A_6\xi^2 - 9A_2A_4A_6^2 + 27A_0A_6^3 + 2A_4^3A_6 - 3\sqrt{3}A_6^2\zeta + 3A_2A_6^2\xi$$
$$- A_4^2A_6\xi + i\sqrt{3}(9A_2A_4A_6^2 - 27A_0A_6^3 - 2A_4^3A_6 + 3\sqrt{3}A_6^2\zeta + 3A_2A_6^2 - A_4^2A_6\xi)]^{1/2},$$

$$k_5 = \frac{1}{\sqrt{3A_6\xi}}[-A_4A_6\xi^2 - 9A_2A_4A_6^2 + 27A_0A_6^3 + 2A_4^3A_6 - 3\sqrt{3}A_6^2\zeta + 3A_2A_6^2\xi$$
$$- A_4^2A_6\xi - i\sqrt{3}(9A_2A_4A_6^2 - 27A_0A_6^3 - 2A_4^3A_6 + 3\sqrt{3}A_6^2\zeta + 3A_2A_6^2 - A_4^2A_6\xi)]^{1/2},$$

$$k_6 = -\frac{1}{\sqrt{3A_6\xi}}[-A_4A_6\xi^2 - 9A_2A_4A_6^2 + 27A_0A_6^3 + 2A_4^3A_6 - 3\sqrt{3}A_6^2\zeta + 3A_2A_6^2\xi$$
$$- A_4^2A_6\xi - i\sqrt{3}(9A_2A_4A_6^2 - 27A_0A_6^3 - 2A_4^3A_6 + 3\sqrt{3}A_6^2\zeta + 3A_2A_6^2 - A_4^2A_6\xi)]^{1/2}$$

and

$$\zeta = (4A_2^3A_6 - A_2^2A_4^2 - 18A_0A_2A_4A_6 + 27A_0^2A_6^2 + 4A_0A_4^3)^{1/2},$$

$$\xi = (36A_2A_4A_6 - 108A_0A_6^2 - 8A_4^3 + 12\sqrt{3}A_6\zeta)^{1/3}.$$

5. RESULTS AND DISCUSSIONS

An intelligent beam structure containing a distributed piezoelectric sensor/actuator on both the top and bottom surface is shown in Figure 3. In this structure, the piezoelectric of the bottom layer is considered as a sensor to sense the strain and generate the electrical potential and the piezoelectric of the top layer as an actuator to control the vibration of the structure. All material properties used are shown in Table 1.

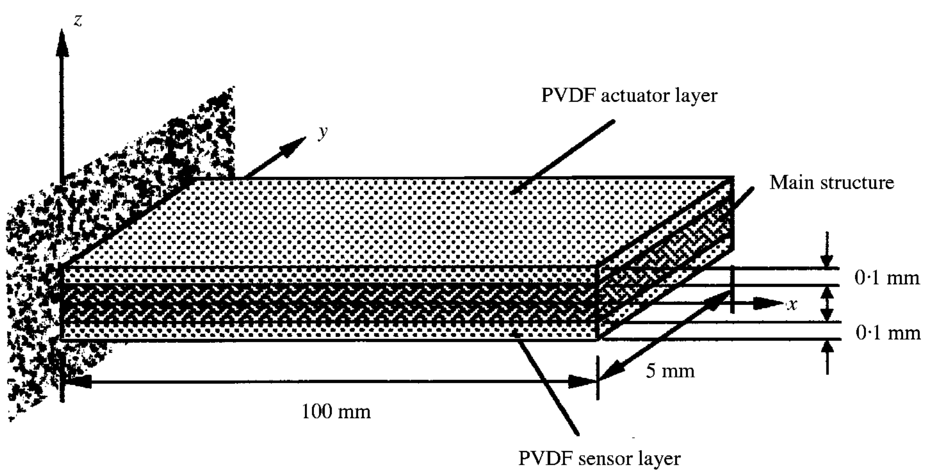

Figure 3. A beam with piezoelectric sensor and actuator.

TABLE 1

The material properties of the main structure and piezoelectric

Property	PVDF	Graphite/epoxy
E_1	0·2E + 10 N/m^2	0·98E + 11 N/m^2
E_2	0·2E + 10 N/m^2	0·79E + 10 N/m^2
G_{12}	0·775E + 9 N/m^2	0·56E + 10 N/m^2
G_{23}	—	0·385E + 10 N/m^2
v_{12}	0·29	0·28
ρ	1800 kg/m^3	1520 kg/m^3
e_{31}	0·046 C/m^2	—
e_{32}	0·046 C/m^2	—
e_{33}	0·0	—
g_{11}	0·1062E − 9 F/m	—
g_{22}	0·1062E − 9 F/m	—
g_{33}	0·1062E − 9 F/m	—
t	0·1E − 3 m	0·125E − 3 m

A cantilever laminated beam with a distributed piezoelectric PVDF layer serving as a distributed actuator on the top surface, and another PVDF on the bottom surface as a distributed sensor, will be used as a case study. The beam dimensions considered are *length* $l = 100$ mm and *width* $b = 5$ mm. The thickness of the beam can be generally written as $h = n \times 0{\cdot}125\mathrm{E} - 3$ m and piezoelectric PVDF layer is taken as $0{\cdot}1 \times 10^{-3}$ m (see Table 1). The applied transverse load is uniformly distributed and has a magnitude of $p_1(x) = p_2(x) = 2{\cdot}5 \times 10^3\,\mathrm{N/m^2}$. Here, all the graphical outputs are obtained by using Software Package MATHEMATICA. The transverse displacement of four-layer laminated composite beams with an actuator and sensor layer on the top and bottom surfaces respectively, for feedback gains of 0, 40, 100 and 140, are shown in the following figures. In the following graphical results, the frequency of the external applied force is taken as 10 Hz.

Figure 4 shows the effect of negative velocity feedback control gain on the tip transient response. The effect of ply orientation on the beam response is studied in Figure 5. It is evident from the graphs that the transient tip amplitude of the beam is damped out quickly when the higher feedback control gains are applied. This also illustrates the applicability of the present approximate solution. From Figure 5, the significant effect of the lamination scheme or stacking sequences of laminated beams can be easily seen.

Figures 6 and 7 present the output and input voltage of vibration of smart laminated beams. Please note that Figures 6 and 7 are based on the feedback control gain $G_i = 40$ C/A with [0/90/90/0°] ply orientation. From these two figures, it can be seen that the output and input voltage vary as the beam vibrates, and their vibrational period is the same as the period of laminated beams. It can also be seen that there is about a $\pi/2$ phasic difference between input and output voltage. Figure 8 shows the tip deflection of the beam versus feedback control gain for the different ply orientations. From this figure, the tip deflection (amplitude) of the beam can be shown to decrease quickly while the feedback control gain increases. When the control gain G_i is less than 100 C/A, the control purpose is very

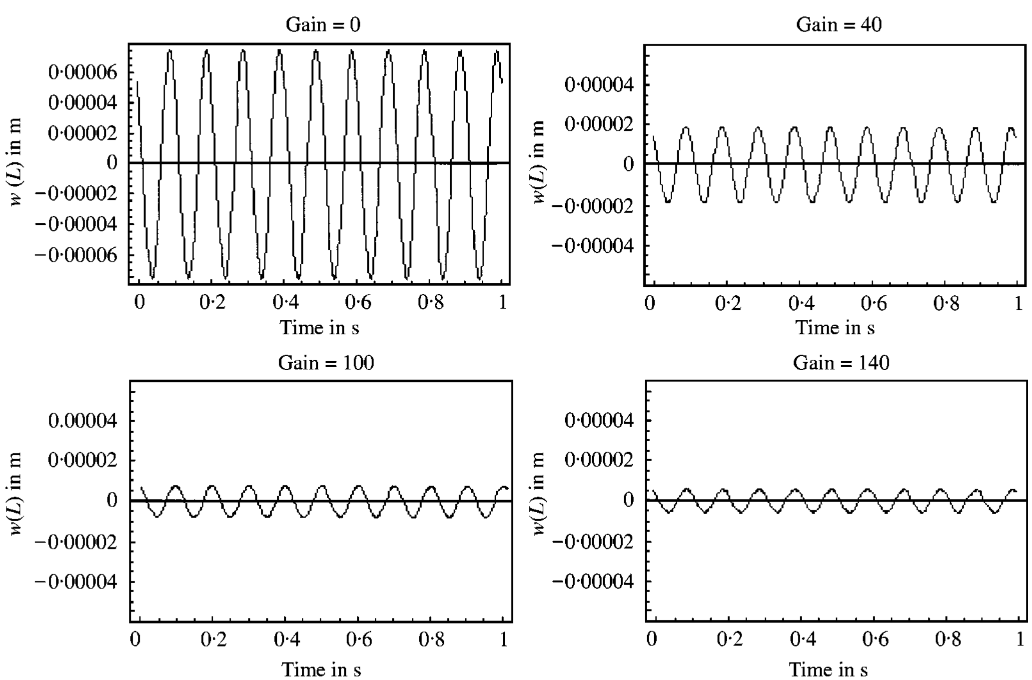

Figure 4. Effect of negative velocity feedback gain on the tip transient response [0/90/90/0°].

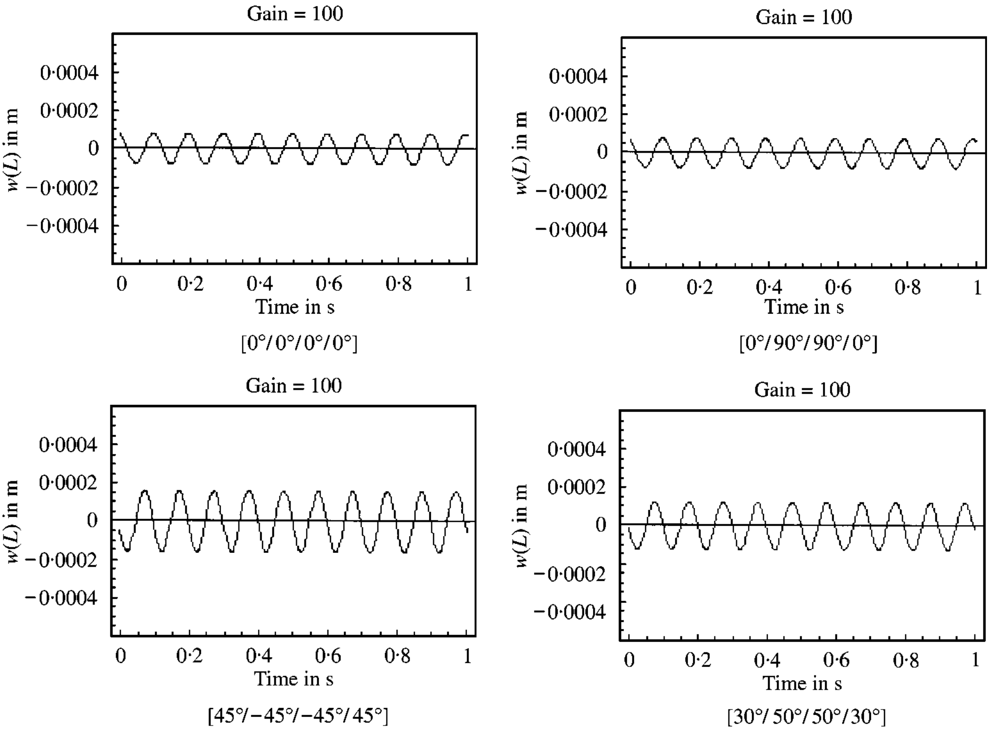

Figure 5. Effect of ply orientation on the transient response.

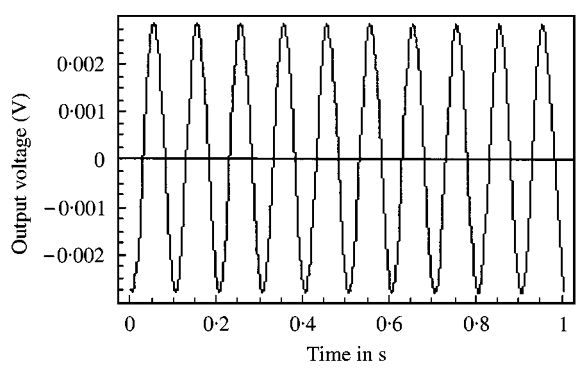

Figure 6. The output voltage from sensor layer.

effective. However, when the control gain $G_i > 100$ C/A, the tip deflection decreases very slowly. From all these phenomena, it can be said that the optimal feedback control gain of the present beam model is about 100 C/A.

6. CONCLUSIONS

An analytical solution for the analysis of laminated composite beams with a piezoelectric sensor and actuator has been presented in this paper. The governing equation of the smart

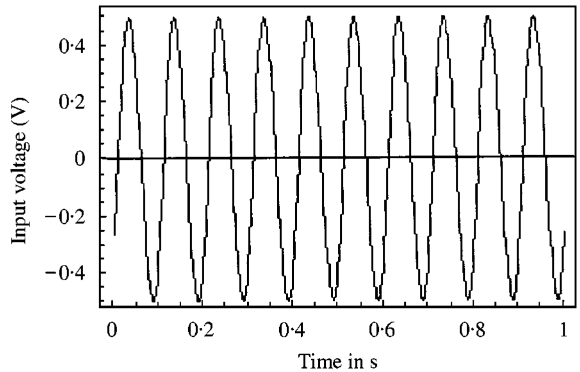

Figure 7. The input voltage on actuator layer.

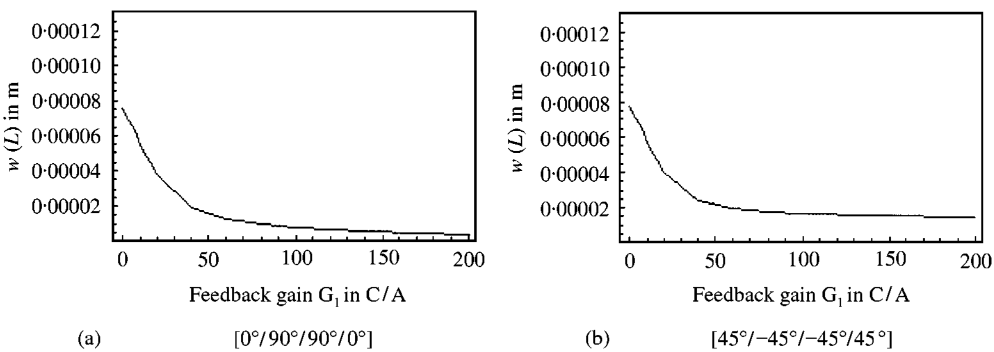

Figure 8. The tip deflection of beam versus feedback control gain.

laminated composite beam based on the Mindlin plate theory has been derived by introducing the electric potential function. The new assumption of the harmonic vibration, which includes the sine and cosine terms, has been also presented. As another contribution, the present method creatively introduced the transformation method of complex numbers to reduce the two-coupled differential equations to one complex differential equation. From numerical results of the dynamic response and analysis of the state governing equations for smart laminated beams, it is observed that the displacement decays amplitude while the feedback gain increases. It is concluded that the present method is correct and effective.

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